Detection of Different Shapes of Lactation Curve for Milk Yield in Dairy Cattle by Empirical Mathematical Models

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ABSTRACT

The study of relationships between mathematical properties of functions used to model lactation curves is usually limited to the evaluation of the goodness of fit. Problems related to the existence of different lactation curve shapes are usually neglected or solved drastically by considering shapes markedly different from the standard as biologically atypical. A deeper investigation could yield useful indications for developing technical tools aimed at modifying the lactation curve in a desirable fashion. Relationships between mathematical properties and lactation curve shapes were analyzed by fitting several common functions (Wood incomplete gamma, Wilmink's exponential, Ali and Schaeffer's polynomial regression, and fifth-order Legendre polynomials) to 229,518 test-day records belonging to 27,837 lactations of Italian Simmental cows. Among the best fits (adjusted r^2 higher than 0.75), the 3-parameter models (Wood and Wilmink) were able to detect 2 main groups of curve shape: standard and atypical. Five-parameter models (Ali and Schaeffer function and the Legendre polynomials) were able to recognize a larger number of curve shapes. The higher flexibility of 5-parameter models was accompanied by increased sensitivity to local random variation as evidenced by the bias in estimated test-day yields at the beginning and end of lactation (border effect). Meaning of parameters, range of their values and of their (co)variances are clearly different among groups of curves. Our results suggest that analysis based on comparisons between parameter values and (co)variances should be done carefully. Comparisons among parameter values and (co)variances could yield more robust, reliable, and easy to interpret results if performed within groups based on curve shape.

(Key words: lactation curve shape, mathematical model, parameter estimate)

Abbreviation key: ADJRSQ = adjusted r^2 , AS = Ali and Schaeffer polynomial regression, LEG = Legendre orthogonal polynomials, TD = test day, WD = Wood incomplete gamma function, WIL = Wilmink exponential function.

INTRODUCTION

Empirical mathematical models of lactation curves are regular functions $y = f(t)$, defined for positive values of daily milk production (y) and time from parturition (t), used in the dairy cattle industry for breeding and management purposes. These models represent an essential research tool for developing and validating mechanistic models, aimed at explaining the main features of the milk production pattern in terms of the known biology of the mammary gland during pregnancy and lactation (Neal and Thornley, 1983).

When fitted to milk test-day (TD) data, an empirical model must be able to disentangle the continuous component from temporary environmental perturbations and to make predictions on milk yields. Several mathematical functions have been proposed (Beever et al., 1991; Sherchand et al., 1995; Grossman and Koops, 1999), differing mainly in the type of regression (linear or nonlinear), in the number of parameters and in their degree of relationships with the main features of a typical lactation pattern, such as peak yield, time at peak, and persistency.

The analysis of relationships between mathematical properties of models and lactation patterns has been focused mainly on the evaluation of fitting performances. Curve modeling usually deals with data of homogeneous groups of animals, and almost all proposed functions are able to fit average patterns with a higher level of accuracy. Individual patterns are also of interest for many practical purposes, e.g., health monitoring, individual feeding, and especially in recent years, genetic evaluations. In this case, due to the effect of several environmental and genetic factors that result in a random variation of shapes between cows, a large range of goodness of fit has been reported by several authors (Wood, 1969; Perochon et al., 1996; Olori et al., 1999).

Received June 30, 2004.

Accepted October 21, 2004.

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Table 1. Means and standard deviations (kg/d) for milk test day yields recorded at different stages of lactation for the 6 parities.

Number of the test	No. of lactations	DIM		Milk yield	
		Mean	SD	Mean	SD
Test 1	27,837	23	13	23.16	6.09
Test 2	27,837	57	15	22.77	6.33
Test 3	27,837	90	16	21.23	6.11
Test 4	27,837	123	18	19.78	5.84
Test 5	27,837	157	19	18.37	5.59
Test 6	27,837	191	20	16.88	5.39
Test 7	27,837	225	21	15.07	5.29
Test 8	20,760	255	20	13.76	5.13
Test 9	10,895	282	19	12.64	4.98
Test 10	3004	305	15	11.84	4.78

There is, however, another important aspect of milk yield lactation curve modeling that has been scarcely considered in literature. Models are usually chosen on the basis of their ability to describe a specific pattern on the plane (t, y), characterized by a first ascending phase to a peak followed by a decreasing phase, i.e., the standard form of the lactation curve. However, they are able to analytically represent several other shapes. This is obvious for general functions such as polynomials, but it is also valid for equations specifically conceived to model the lactation curve (e.g., Wood and Wilmink functions). This feature is useful, for example, when other milk production traits (fat and protein contents) or data coming from other species (Lan-dete-Castillejos and Gallego, 2000) are modeled with the same function, but can bring about complications when different patterns occur for the same trait, as is the case of milk yield. In fact, several shapes of lactation curves are detected when milk test-day data are fitted with empirical regression equations. Some of them consist of slight modifications of the standard curve as, for example, the presence/absence of an inflection point in the decreasing part of lactation (Druet et al., 2003), whereas others are markedly different as in the case of continuously decreasing curves that lack lactation peak (Congleton and Everett, 1980; Shanks et al., 1981; Olori et al., 1999).

Two main issues arise from the existence of such a polymorphism. The first (more theoretical) question is to assess whether the occurrence of different shapes has biological basis or is the result of random perturbations (missing records, outliers). A more technical aspect is related to the eventual differences both in the range of parameter values and in their mathematical interpretation when the same function is fitted to curves with different shapes.

The present study is aimed at achieving a better understanding of the mathematical and biological nature of the different shapes of milk yield lactation

curves by checking the different shapes of lactation curves detected by some of the most commonly used functions, and by analyzing relationships between lactation curve shape and characteristics of function parameters.

MATERIALS AND METHODS

Data

An archive of 229,518 TD records belonging to 27,837 lactations of 16,732 cows, recorded from 1989 to 2002, was extracted from the historical data set of the Italian Association of Simmental Cow Breeders. Edits were on the number of TD records per cow (>6), parity (<6), testing scheme (A4), lactation length (<340 d), and DIM at which the first TD was recorded (>5). Average milk yields and DIM for each test are reported in Table 1.

Mathematical Functions

The following functions were chosen to model the lactation curve: the incomplete gamma function of Wood (Wood, 1967), the Wilmink's function (Wilmink, 1987), the 5-parameter polynomial regression of Ali

Table 2. Theoretical curve shapes that can be fitted by Wood and Wilmink functions (the parameter a is always greater than 0).

Model	Parameter		Curve shape
	b	c	
Wood	+	–	Standard curve
	–	–	Continuously decreasing (atypical) curve
	–	+	Reversed standard
	+	+	Continuously increasing
Wilmink	–	–	Standard curve
	+	–	Continuously decreasing (atypical) curve
	+	+	Reversed standard
	–	+	Continuously increasing

Table 3. Absolute and relative (below, in *italics*) frequencies of fits among different classes of adjusted r-square (ADJRSQ).

ADJRSQ class	Model			
	Wood	Wilink	Ali and Schaeffer	Legendre
<0.20	1551 <i>5.5</i>	1664 <i>6.0</i>	1095 <i>3.9</i>	1139 <i>4.1</i>
0.20 to 0.40	1288 <i>4.6</i>	1397 <i>5.0</i>	855 <i>3.1</i>	861 <i>3.1</i>
0.40 to 0.60	2876 <i>10.3</i>	2739 <i>9.8</i>	1709 <i>6.1</i>	1826 <i>6.6</i>
0.60 to 0.80	6919 <i>24.9</i>	6354 <i>22.8</i>	4488 <i>16.1</i>	4519 <i>16.2</i>
>0.80	15,203 <i>54.6</i>	15,683 <i>56.3</i>	19,690 <i>70.7</i>	19,492 <i>70.0</i>

and Schaeffer (Ali and Schaeffer, 1987), and a fourth-order normalized Legendre polynomial (Kirkpatrick et al., 1990).

Let Y_t denote the milk test-day yield at time t (in d) from parturition. The Wood model (**WD**)

$$Y_t = at^b e^{ct}$$

fitted in the log linear form

$$\log Y_t = \log a + b \log t + ct \quad [1]$$

has the advantage of having a limited number of parameters with an assessed direct reference to main features of lactation curve (Rekaya et al., 2000). It is still one of the most frequently used functions.

The combined exponential and linear model proposed by Wilink (**WIL**)

$$Y_t = a + be^{-kt} + ct \quad [2]$$

can be reduced to a 3-parameter linear model by setting the k exponent to a suitable fixed value. In this form, it is considered the best 3-parameter model of the lactation curve (Olori et al., 1999) and it is used

in random regression analysis (Schaeffer et al., 2000). In the present study, k was assumed equal to 0.10 (Brotherstone et al., 2000). Moreover, WIL parameters can also be easily related to the characteristics of the lactation curve shape.

The Ali and Schaeffer polynomial regression (**AS**)

$$Y_t = a + b(t/340) + c(t/340)^2 + d \log(340/t) + k[\log(340/t)]^2 \quad [3]$$

and the Legendre orthogonal polynomials (**LEG**)

$$Y_t = \alpha_0 \times P_0 + \alpha_1 \times P_1 + \alpha_2 \times P_2 + \alpha_3 \times P_3 + \alpha_4 \times P_4 \quad [4]$$

are frequently used in random regression and covariance function models. These 2 functions have been selected to compare differences in model flexibility due to the number of parameters and the degree of correlation among them. Functions of time (P_j) of LEG model were calculated using values published by Schaeffer (2004).

Table 4. Means and standard deviations of parameters for individual curves classified according to the 4 shapes detected by the Wood model.

Shape ¹	Frequency	Log a		b		c	
		Mean	SD	Mean	SD	Mean	SD
1	14,181	2.43	0.85	0.267	0.236	-0.00564	0.00286
2	3055	3.57	0.42	-0.104	0.094	-0.00204	0.00122
3	455	4.59	1.08	-0.430	0.281	0.00143	0.00186
4	1	2.57		0.012		0.00122	

¹1 = Standard curve, 2 = continuously decreasing (atypical) curve, 3 = reversed standard, 4 = continuously increasing.

Table 5. Means and standard deviations of parameters for individual curves classified according to the 4 shapes detected by the Wilmink model.

Shape ¹	Frequency	a		b		c	
		Mean	SD	Mean	SD	Mean	SD
1	11,542	29.15	7.27	-81.89	364.85	-0.0636	0.0247
2	6412	23.75	6.20	102.89	389.72	-0.0466	0.0205
3	33	12.91	5.76	545.67	930.45	0.0121	0.0102
4	10	20.87	5.77	-23.32	17.44	0.0073	0.0057

¹1 = Standard curve, 2 = continuously decreasing (atypical) curve, 3 = reversed standard, 4 = continuously increasing.

Model Properties

The ability of the 3-parameter functions to recognize lactation curves with different shapes can be tested based on their analytical properties that are developed from first/second derivatives (see Appendix).

A relative maximum (or minimum) in the Wood function can be found at $t = -\frac{b}{c}$, where the first derivative equals zero. As a consequence, t being positive, accepted solutions are

$$b > 0 \text{ and } c < 0 \text{ or}$$

$$b < 0 \text{ and } c > 0.$$

Because the second derivative is negative at maximum, the solution $b > 0$ and $c < 0$ corresponds to the standard form of the lactation curve. On the contrary, the solution $b < 0$ and $c > 0$ corresponds to a reversed shape, with an initial decreasing phase to a minimum, followed by an increase. The combination of $b > 0$ and $c > 0$ represents a continuously increasing curve, whereas the solution $b < 0$ and $c < 0$ describes a continuously decreasing curve, i.e., the so-called atypical curve for milk yield (Congleton and Everett, 1980).

The same 4 types of curves can be described by the WIL model [2].

A relative maximum (or minimum), for $t > 0$, can be found at

$$t = -\frac{1}{k} \log \left(\frac{c}{kb} \right)$$

Table 6. Correlation matrix of Wood function parameters in the 2 main curve shapes.

Parameter	Standard		Atypical	
	b	c	b	c
Log a	-0.95	0.73	-0.80	0.21
b		-0.84		-0.40

provided that

$$0 < \frac{c}{kb} < 1$$

Therefore it can be concluded that if $b < 0$ and $c < 0$, the WIL model describes the standard form of the lactation curve, whereas if $b > 0$ and $c > 0$ the curve is reversed with a point of minimum. Finally, $b < 0$ and $c > 0$ results in a continuously increasing curve, whereas $b > 0$ and $c < 0$ corresponds to a continuously decreasing curve.

The analytical study of WD and WIL models is summarized in Table 2.

The mathematical study of AS and LEG functions is less straightforward. However, the different shapes they are able to mathematically describe can be deduced likewise on the basis of the possible combinations of parameter signs; thus, both models can theoretically describe 2⁵ forms.

Lactation Curve Fitting

The first step for evaluating the lactation curve shapes actually found is to discharge curves that originate from a poor fit. Indeed the main aim of the mathematical modeling of lactation curves is the detection of the component of variability that can be ascribed to the evolution of milk production over time along lactation. Consequently, poor fitting performances can be due either to a wrong choice of the function used or to a marked predominance of random perturbation over the continuous and regular component. In the present study, the 4 models considered were fitted to TD records of each individual lactation, goodness of fit was assessed by the adjusted R-square (**ADJRSQ**) and curves were classified according to 5 levels of ADJRSQ (1 < 0.20, 2 = 0.20 to 0.40, 3 = from 0.40 to 0.60, 4 = from 0.60 to 0.80, 5 = >0.80). For each model, individual curves with ADJRSQ greater than 0.75 were grouped according to the different combinations of parameter signs, each corresponding to a particular shape. Finally, a detailed analysis was carried out

Figure 1. Examples of individual curves of standard shape detected by the Wood model selected for increasing values of the b parameter ($\blacksquare = 0.0317$, $\blacktriangle = 0.267$, $\blacklozenge = 0.537$)

within the larger groups to assess: 1) relationships between lactation curve shapes and parameter values and (co)variances; and 2) meaning of function parameters in the different groups.

RESULTS AND DISCUSSION

Goodness of fit was quite high for all the models and it increased, as expected, with the number of function parameters (Table 3). A large proportion of curves showed an ADJRSQ higher than 0.8, in agreement with previous results reported for dairy cattle (Olori et al., 1999).

All of the models were able to recognize different shapes among curves with best fits ($\text{ADJRSQ} > 0.75$). The 3-parameter models classified individual lactation shapes essentially into 2 main groups (Tables 4 and 5),

Figure 2. Examples of individual curves of atypical shape detected by the Wood model selected for increasing values of the b parameter ($\blacksquare = 0.202$, $\blacktriangle = -0.105$, $\blacklozenge = -0.001$).

Figure 3. Decomposition of Wilmink function in standard curves ($\blacksquare = a$; $\blacktriangle = b^{-kt}$; $\blacklozenge = ct$).

corresponding to standard and atypical (continuously decreasing) lactation curves respectively. The WD model was able to recognize a higher number of standard patterns in comparison with the WIL model (80 vs. 64% respectively).

Correlations among WD curve parameters show the same signs in both of the 2 main groups (Table 6), whereas absolute values were lower in atypical curves. Several authors have pointed out contradictions in signs of correlations, both phenotypic and genetic, between WD function parameters (Shanks et al., 1981; Ferris et al., 1985; Varona et al., 1998; Rekaya et al., 2000; Tekerly et al., 2000), especially those regarding the parameter c . However, this is explained by the fact that the authors did not use exactly the same function (they changed the sign of c).

According to the second derivative of Wood function [5], the absolute value of b parameter controls the magnitude of the curvature of the lactation pattern,

Figure 4. Decomposition of Wilmink function in atypical curves ($\blacksquare = a$; $\blacktriangle = b^{-kt}$; $\blacklozenge = ct$).

Table 7. Correlation matrix of Wilmink function parameters in the 2 main curve shapes.

Parameter	Standard		Atypical	
	b	c	b	c
a	-0.11	-0.76	-0.02	-0.74
b		0.09		-0.06

i.e., its deviation from a straight line (Congleton and Everett, 1980). Moreover, the curve is concave when [5] is negative (i.e., b is positive) and convex when [5] is positive (i.e., b is negative). These properties are highlighted in Figures 1 and 2, which report examples of individual standard and atypical curves respectively, selected for increasing values of the b parameter. In standard curves (Figure 1), for larger values of b , there is a more rapid rate of increase of estimated yields in the first part of lactation, whereas in atypical curves (Figure 2), increasing values of b result in a reduced rate of decline in this phase. Moreover, parameter variances and correlations (Tables 4 and 6) show higher absolute values for standard curves, resulting in a larger impact of b variations on the c parameter. These figures underline differences in the meaning of b parameter between the 2 groups of curves. These differences are enhanced by the occurrence of high values, both positive and negative, of the parameter b evidenced by its large standard deviation in both groups (Table 4). All these results suggest that comparing parameter values across different groups must be done carefully.

Similar considerations can be developed for the WIL model. According to the second derivative [6], the absolute value and the sign of parameter b controls the

Figure 5. Examples of individual curves of standard shape detected by the Wilmink model selected for increasing values of b ($\blacksquare = -441.21$, $\blacktriangle = -81.99$, $\blacklozenge = -0.91$). **Figure 6.** Examples of individual curves of atypical shape detected by the Wilmink model selected for increasing values of b ($\blacksquare = 0.0053$, $\blacktriangle = 103.56$, $\blacklozenge = 489.92$).

magnitude and the type of curvature, respectively. A better understanding of WIL parameters in the 2 main groups of curves can be achieved by analyzing the 3 additive components of the function (Figures 3 and 4). The constant term a is the scaling factor in both groups. The exponential term $b^{(kt)}$ increases in standard curves (Figure 3) and decreases in atypical curves (Figure 4). Thus the parameter b , being the asymptotic value of this exponential term for $t = 0$, controls the rate of variation of milk yield in the first part of the curve, even if indirectly: higher absolute values of b result in faster increasing (or decreasing) rates. Finally, the c parameter, which measures the slope of the straight line ct , is mainly related to the rate of decline in the second part of lactation, i.e., the lactation

Figure 7. Pattern of variability of Wilmink b parameter for different intervals at which the first test day occurs.

Table 8. Distribution of standard and atypical lactation curves for milk yield (absolute and *relative* frequencies) for Wood (WD) and Wilmink (WIL) model across different classes of distance of the first test from calving.

Model	Curve shape	Distance of the first test from calving (d)		
		5 to 30	30 to 60	>60
WD	Standard	10,795 <i>0.76</i>	3162 <i>0.22</i>	224 <i>0.02</i>
	Atypical	2141 <i>0.70</i>	869 <i>0.28</i>	45 <i>0.02</i>
WIL	Standard	9125 <i>0.79</i>	2252 <i>0.20</i>	165 <i>0.01</i>
	Atypical	4261 <i>0.66</i>	2027 <i>0.32</i>	124 <i>0.02</i>

persistence. A distinguishing feature of WIL model in comparison with WD is the substantial independence between the first and the second part of the curve, as evidenced by the low correlations between *b* and *c* for both standard and atypical curves (Table 7). Such a different degree of dependence among parameters could be the reason for the larger number of standard curves detected by the WD model. Differences in the ability of models to describe different lactation curve shapes have been reported by Landete-Castillejos and Gallego (2000).

Figures 5 and 6 report some examples of individual standard and atypical curves, respectively, selected for increasing values of the WIL *b* parameter. These patterns confirm the relationship between *b* values and rate of variation in the first part the curve. Moreover, a clear bias in estimates of TD milk yields in the first days of lactation can be observed for high absolute values of *b* in both groups. In fact, high absolute values of *b* are quite common, as evidenced by the large standard deviation of this parameter in the 2 groups (Table 6). The large variability of *b* is mainly related to the

time distance between calving and the first test: a sudden increase of the standard deviation of *b* occurs when the first test is at more than 30 d from calving (Figure 7). All these results suggest that, also in WIL model, a correct comparison of parameter values should be done within groups, i.e., for curves showing the same shape.

A deeper understanding of parameter significance, both for WD and WIL models, may add some interesting elements to the discussion on the occurrence of different shapes of lactation curve for milk yield. Several authors state that the occurrence of curves without lactation peak is mainly a result of the absence of records in the first days of lactation. For example, a relevant occurrence of negative values of WD *b* parameter has been reported for lactations with the first test 30 d or more after freshening (Congleton and Everett, 1980). Results of this study reported in Table 8 confirm that the occurrence of atypical patterns is more a mathematical issue that arises from the date at first test but also from the peculiar combinations of TD values and their distribution along the whole lactation

Table 9. Distribution of standard and atypical lactation curves for milk yield (absolute and *relative* frequencies) fitted by the Wilmink (WIL) model across different parities and calving seasons.

Parity	Curve shape		Calving season	Curve shape	
	Standard	Atypical		Standard	Atypical
First	2971 <i>0.67</i>	1425 <i>0.33</i>	January–February	2757 <i>0.73</i>	997 <i>0.27</i>
Second	3297 <i>0.63</i>	1920 <i>0.37</i>	March–April	2127 <i>0.68</i>	1019 <i>0.32</i>
Third	2465 <i>0.63</i>	1452 <i>0.37</i>	May–June	1309 <i>0.59</i>	914 <i>0.41</i>
Fourth	1688 <i>0.63</i>	990 <i>0.37</i>	July–August	1130 <i>0.51</i>	1111 <i>0.49</i>
Fifth	1121 <i>0.64</i>	625 <i>0.36</i>	September–October	1739 <i>0.60</i>	1171 <i>0.40</i>
			November–December	2480 <i>0.67</i>	1200 <i>0.33</i>

Table 10. Absolute frequencies of the groups¹ of curves detected by the Ali and Schaeffer model.

Groups	a	b	c	d	k	Frequency (n)
1	–	+	–	+	–	12,087
2	+	–	+	–	+	6836
3	+	–	–	–	+	674
4	+	+	–	+	–	445
5	+	–	+	+	–	333
6	+	+	–	–	+	206
7	–	+	+	+	–	200
8	+	–	+	–	–	141
9	+	–	–	+	–	120
10	–	+	–	+	+	72
11	+	+	–	+	+	65
12	+	+	+	+	–	35
13	–	–	+	+	–	35
14	+	–	–	–	–	30
15	+	+	–	–	–	9
16	+	–	+	+	+	7
17	+	–	–	+	+	7
18	–	+	–	–	+	5
19	+	+	+	+	+	1

¹Each group corresponds to a different combination of parameter signs.**Table 11.** Means and standard deviations of parameters of individual curves classified according to some of the groups¹ detected by the Ali and Schaeffer model.

Group	Frequency	a	b	c	d	k
		Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)
1	12,087	–259.87 (303.062)	426.25 (444.85)	–170.19 (158.28)	–176.24 (197.96)	–30.60 (37.70)
2	6836	293.16 (341.65)	–419.17 (489.45)	142.11 (162.75)	–167.49 (226.67)	29.72 (44.47)
3	674	65.88 (36.56)	–46.51 (39.40)	–18.10 (16.18)	–31.17 (27.92)	6.57 (6.73)
4	445	8.71 (6.61)	22.28 (15.95)	–26.54 (19.52)	10.60 (5.31)	–1.86 (1.26)
5	333	20.81 (11.63)	–31.02 (24.98)	23.15 (21.59)	9.45 (7.71)	–2.74 (2.49)
6	206	25.74 (14.08)	23.91 (22.22)	–51.26 (28.63)	–11.16 (12.18)	–3.96 (3.99)
7	200	–33.22 (33.00)	35.00 (32.18)	14.13 (15.71)	45.44 (28.45)	–9.54 (6.79)

¹Each group corresponds to a different combination of parameter signs.**Table 12.** Correlation matrix of Ali and Schaeffer function parameters in the 2 main groups of curves.¹

Parameter	Group 1				Group 2			
	b	c	d	k	b	c	d	k
a	–0.99	0.93	–0.99	0.98	–0.99	0.93	–0.99	0.98
b		–0.96	0.98	–0.96		–0.96	0.98	–0.96
c			–0.91	0.87			–0.91	0.86
d				–0.99				–0.99

¹Each group corresponds to a different combination of parameter signs.

Table 13. Correlation matrix of Ali and Schaeffer parameters in groups¹ 3 and 4 reported in Figure 6.

Parameter	Group 3				Group 4			
	b	c	d	k	b	c	d	k
a	-0.30	-0.09	-0.45	0.21	-0.94	-0.19	-0.95	0.87
b		-0.85	0.01	0.17		-0.10	0.87	-0.75
c			0.23	-0.32			-0.26	-0.31
d				-0.91				-0.97

¹Each group corresponds to a different combination of parameter signs.

length. These combinations determine the sign of the parameter *b* and the shape, standard or atypical, of the lactation curve. Therefore, when tests are concentrated in the declining phase of lactation, if daily yields can be fitted better by a concave curve, then an increasing phase to a maximum is extrapolated and the curve assumes a standard shape. Such a consideration is supported by the distribution of curves in both WD and WIL models for different classes of distance between calving and day at the first test (Table 8). About 20% of standard curves had the first test at 30 or more days after calving. However, a role of the biological differences among cows should not be excluded. An effect of calving season and parity on the occurrence of atypical curves has been reported (Rekik and Ben Gara, 2004). Results of our study (Table 9) indicate differences of the distribution of the 2 types of shapes across calving seasons, with summer calvings showing an higher occurrence of atypical patterns, in agreement with the above cited study.

As expected, 5-parameter models were able to recognize a larger number of groups of curves. Nineteen and 18 out of the 32 theoretical groups were detected by AS and LEG, respectively. However, each group of curves can be considered the result of a specific

deformation of the 2 basic shapes, standard or atypical, that are made more variable by the presence of waves in the second part of lactation.

The main difference between the 2 large groups detected by the AS model is the opposite combination of parameter signs (Table 10). Averages of absolute values, variances (Table 11), and correlations (Table 12) are similar. Parameter values are markedly larger than those reported in literature (Ali and Schaeffer, 1987; Olori et al., 1999); however, cited studies deal with average curves, whereas our work refers to averages of individual parameters. Correlations among parameters are very large. This is a well-known limitation of the AS model (Kettunen et al., 2000), which can hinder the estimation process, especially in genetic analysis. The AS function has been largely abandoned in favor of orthogonal polynomials (Schaeffer, 2004).

The other groups of curves differ in the sign of some parameters and have smaller average parameter values and variances (Tables 11 and 13).

Figures 8, 9, 10, and 11 show examples of estimated individual curves of four different groups. For brevity, curves have been selected from the 4 large groups (1, 2, 3, and 4 of Table 11), and for increasing values of the coefficient of the linear term (*b*) of equation [3].

Figure 8. Examples of individual curves of group 1 of Table 10 detected by the Ali and Schaeffer model selected for increasing values of parameter *b* (■ = 100.84; ▲ = 426.82; ◆ = 843.89).

Figure 9. Examples of individual curves of group 1 of Table 10 detected by the Ali and Schaeffer model selected for increasing values of parameter *b* (■ = -843.66; ▲ = -420.21; ◆ = -120.33).

Figure 10. Example of individual curves of group 3 of Table 10 detected by the Ali and Schaeffer model selected for increasing values of parameter b (■ = -89.36; ▲ = -46.39; ◆ = -6.10).

In group 1 (Figure 8), the increase of b results in a large, increasing rate of milk yield at the beginning of lactation, whereas in group 2 an increase in b enhances the rate of initial decline of the curve (Figure 9). In both groups, the presence of 2 points of inflection can be observed. This is a clear consequence of the higher flexibility of 5-parameter models that are able to account for differences in rate of increase or decrease in milk yield along the entire lactation. Moreover, due to the opposite parameter signs, sequences of changes in curvature are opposite in 2 groups (concave-convex-concave for group 1 and convex-concave-convex for group 2). Finally, the high correlation among all parameters (Table 12) results in an enhancement of curvatures for increasing values of b .

Lactation patterns extracted from groups 3 and 4 (Figures 10 and 11) are examples of standard and atyp-

Figure 11. Example of individual curves of group 4 of Table 10 detected by the Ali and Schaeffer model selected for increasing values of parameter b (■ = 6.13; ▲ = 22.36; ◆ = 38.37).

Table 14. Absolute frequencies of the groups¹ of curves detected by the Legendre orthogonal polynomials.

Group	α_0	α_1	α_2	α_3	α_4	Frequency (n)
1	+	-	-	-	-	9600
2	+	-	+	+	+	2566
3	+	+	+	+	+	1751
4	+	-	-	+	-	1545
5	+	-	-	-	+	1369
6	+	-	-	+	+	1035
7	+	-	+	-	+	1013
8	+	-	+	+	-	790
9	+	-	+	-	-	788
10	-	-	-	-	-	614
11	+	+	+	+	-	50
12	+	+	-	+	-	21
13	+	+	-	+	+	5
14	+	+	+	-	+	4
15	+	+	-	-	-	4
16	+	+	+	-	-	1
17	+	+	-	-	+	1
18	-	-	-	-	+	1

¹Each group corresponds to a different combination of parameter signs.

ical curves respectively. However, some substantial differences in comparison with the 2 main groups can be observed. In particular, standard curves (Figure 10) have a constant curvature (i.e., they lack an inflection point), whereas atypical patterns have an inflection point near the middle of lactation. In both of these groups, the effects of variation in b are less evident than in the 2 main groups, essentially due to the reduced correlations among parameters (Table 13).

Another important difference between groups of large and small size is reflected in the estimation of daily milk yield at the beginning and at the end of the lactation curve. A peculiar border effect, pointed out by several authors (Kirkpatrick et al., 1990; Meyer, 1998; Pool and Meuwissen, 2000), can be clearly observed in the 2 main groups (Figures 8 and 9), with negative or very high estimated TD yields at the edges of the lactation trajectory. Such an effect is reduced or absent in the other 2 small groups.

The two main groups detected by the LEG model show differences in sign only for the last 3 parameters (Table 14). Parameter estimates show similar averages of absolute values and variances across all groups (Table 15). Moreover, variances are limited in magnitude thus restricting the range of individual parameter values in comparison with results of AS model. The 2 main groups show differences in sign only for the last 3 parameters (Table 14). Also, for the LEG model, parameter meaning is related to the group of curves, as evidenced by Figures 12, 13, 14, and 15. These figures report examples of individual curves from groups 1, 2, 4, and 6 of Table 15, selected for increasing values of

Table 15. Distribution of parameters of individual curves classified according to some of the groups¹ detected by the Legendre orthogonal polynomials.

Group	Frequency	α_0	α_1	α_2	α_3	α_4
		Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)
1	9600	14.22 (5.60)	-14.04 (8.28)	-7.43 (6.60)	-4.40 (3.92)	-2.68 (2.09)
2	2566	18.45 (4.80)	-4.71 (2.95)	3.36 (2.73)	2.68 (2.03)	2.00 (1.61)
3	1751	26.43 (12.02)	14.66 (21.01)	19.10 (18.22)	11.71 (9.98)	5.25 (4.16)
4	1545	18.58 (4.85)	-7.39 (3.61)	-2.35 (2.22)	1.07 (1.00)	-1.40 (1.28)
5	1369	16.44 (4.86)	-9.13 (4.50)	-2.61 (2.39)	-1.75 (1.62)	0.80 (0.78)
6	1035	18.47 (5.04)	-7.46 (3.46)	-1.69 (1.53)	1.33 (1.11)	1.05 (0.88)
7	1013	17.05 (4.57)	-6.82 (3.08)	2.26 (2.84)	-1.42 (1.35)	1.64 (1.77)

¹Each group corresponds to a different combination of parameter signs.

the α_1 parameter. In curves of the main group (Figure 12), an increase of the α_1 parameter results in a decrease of the level of the curve and in an enhancement of the curvatures. On the other hand, in curves of group 2, a reduction of both the speed of decrease in the first part of the curve and of the magnitude of curvatures can be observed (Figure 13). Moreover, the succession of curvatures is different in the 2 groups: concave-convex-concave and convex-concave-convex for groups 1 and 2, respectively. In both the 2 groups of smaller size, the increase of α_1 results in a reduction of the level of the curve and of the magnitude of curvatures (Figures 14 and 15).

Figure 12. Examples of individual curves of group 1 of Table 14 detected by the Legendre polynomials selected for increasing values of parameter α_1 (■ = -22.22; ▲ = -14.61; ◆ = -6.86).

Due to mathematical properties of orthogonal polynomials, correlations among LEG parameters are generally lower than those obtained for AS parameters, confirming results reported by other authors (Pool and Meuwissen, 2000). However, some high values (0.91) can be still observed for curves of group 1 (Table 16), whereas correlations for the 2 small groups are negligible (Table 17).

Estimated curves of the LEG model (Figures 11 and 12) also show a border effect with biases at the beginning and, especially, at the end of lactation. The better results of the LEG in comparison with the AS model can be explained with the higher flexibility of orthogo-

Figure 13. Examples of individual curves of group 2 of Table 14 detected by the Legendre polynomials selected for increasing values of parameter α_1 (■ = -7.71; ▲ = -4.71; ◆ = -1.69).

Figure 14. Example of individual curves of group 4 of Table 14 detected by the Legendre polynomials selected for increasing values of parameter α_1 (■ = -10.99; ▲ = -7.40; ◆ = -3.80).

Figure 15. Example of individual curves of group 13 of Table 14 detected by the Legendre polynomials selected for increasing values of parameter α_1 (■ = -11; ▲ = -7.47; ◆ = -4.06).

nal polynomials in comparison with parametric models.

Biological reasons for the large number of groups of curves detected by the AS and LEG models are difficult to find. More likely, such a wide range of shapes and the variation of magnitude of curvatures within each shape arise from the combination of the great flexibility of these functions with the effects of local variations. However, given the size of some of these groups of curves (Tables 10 and 14) and the difficulty in correcting individual lactation patterns for local perturbations, it would be wise to treat them as possible shapes of the lactation curve for milk yield. If this approach is accepted, results of the present study regarding changes in meaning, sign, values and (co)variances of function parameters among different groups suggest that a consistent comparison of individual functions should be made within each group. A relevant example is represented by the random regression analysis of TD data where AS function and LEG polynomials are frequently used to model individual deviations from a fixed average curve (Schaeffer, 2004). The ability of these functions to describe a large number of shapes is particularly useful in this context even if some drawbacks, such as the overestimation of vari-

ances at the edges of the lactation trajectory (Meyer, 1998; Pool et al., 2000), are still present. Results can be improved and made more robust by properly taking into account the differences in parameter meaning and covariances in the different groups of lactation curve shape.

CONCLUSIONS

The mathematical functions proposed to describe the typical milk yield during lactation are able to represent different shapes that may occur when individual patterns are fitted. Our study of the 3-parameter functions, WD and WIL, revealed that different shapes correspond to different combinations of parameter signs. The analysis carried out on lactation patterns that showed an adjusted r^2 higher than 0.75 was able to detect 2 fundamental groups of curves, standard and atypical. Five-parameter models are able to identify a larger number of shapes, but all of them can be considered as subunits of the 2 main types.

The partition of curves among different groups does not seem to have only a biological meaning but it results from the mathematical forcing of the actual pattern into the form (among all those that the model is

Table 16. Correlation matrix of Legendre function parameters in the 2 main groups¹ of curves.

Parameter	Group 1				Group 2			
	α_1	α_2	α_3	α_4	α_1	α_2	α_3	α_4
α_0	0.46	0.49	0.54	0.32	-0.09	0.27	0.39	0.31
α_1		0.91	0.88	0.68		0.32	0.22	0.07
α_2			0.91	0.74			0.53	0.60
α_3				0.74				0.41

¹Each group corresponds to a different combination of parameter signs.

Table 17. Correlation matrix of Legendre function parameters in groups¹ 4 and 13 reported in Figure 8.

Parameter	Group 4				Group 13			
	α_1	α_2	α_3	α_4	α_1	α_2	α_3	α_4
α_0	-0.20	-0.11	0.16	-0.05	-0.09	-0.08	0.23	0.10
α_1		0.21	0.02	0.15		0.39	-0.11	-0.12
α_2			-0.18	0.57			0.03	0.00
α_3				-0.22				0.30

¹Each group corresponds to a different combination of parameter signs.

able to describe) that best fits the combination of values and distribution of test-day records along the lactation trajectory. A clear proof of such a computational artifact is represented by the border effect, particularly evident in the main groups of curves. In the WD and WIL model, the mathematical forcing is essentially due to the single curvature for the entire curve [completely concave (standard shape) or convex (atypical)]. Five-parameter models like AS and LEG allow for more curvatures but with a predefined succession. These functions are therefore more flexible even though such a feature makes them more sensitive to local random variation in milk production.

Results of our study show that meaning of parameters and their (co)variances are clearly different among groups of curves. Care is required when analyses based on comparison between parameter values and (co)variances are performed.

ACKNOWLEDGMENTS

Research was funded by the Italian Ministry of University and Research (grant PRIN ex 40%). Authors wish to acknowledge the 2 anonymous referees for their helpful comments and suggestions.

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$$\frac{d^2Z}{dt^2} = -\frac{b}{t^2} \quad [5]$$

APPENDIX

The first derivative of the Wilmink function is:

For the Wood function, let $\log Y = Z$, the first derivative of the log linear form [1] becomes

$$\frac{dy}{dt} = -kbe^{-kt} + c$$

$$\frac{dZ}{dt} = \frac{b}{t} + c$$

whereas the second derivative is

whereas the second derivative is

$$\frac{d^2y}{dt^2} = k^2be^{-kt} \quad [6]$$