Selecting Mating Pairs with Linear Programming Techniques

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ABSTRACT
Mate selection can increase progeny merit if overall merit is nonlinear for one or more component traits. An index of expected progeny merit could be calculated for all possible mating pairs, and the set of pairs with the highest progeny mean could be selected. There are serious computational problems for more than a few males and females. To select and mate f females, and m males, from n of each, with k0 females per male, would require \((f)(n)f!/(k0!)m\) evaluations. Linear programming algorithms can determine the optimal strategy efficiently by considering only a subset of these possibilities. Let \(\pi_{ij}\) be the index of progeny merit of the \(i^{th}\) sire mated to the \(j^{th}\) dam and \(X_{ij}\) be the decision variable for that mating (restricted to 0 or 1). Then the problem of selecting mating pairs can be stated as: maximize \(\sum_i \sum_j \pi_{ij} X_{ij}\), subject to 1) \(\sum_j X_{ij} < 1\), 2) \(\sum_i X_{ij} < k_0\), 3) \(\sum_i \sum_j X_{ij} = f\), and 4) \(X_{ij} = 0\) or 1. By including an artificial sire and an artificial dam and choosing appropriate merit values for the artificial matings, this problem can be solved by efficient “transportation” algorithms. These algorithms could be used to develop rational mating packages for dairy artificial insemination studs provided that an objective evaluation of progeny merit could be formulated, provided that merit is not simply additively inherited.

INTRODUCTION
Animal breeders usually assume, in livestock selection theory, that selected males and females are mated randomly. Average progeny merit can be increased by mate selection if overall merit is not inherited additively (1). Merit will not be inherited additively if either 1) component traits are not inherited additively or 2) merit is defined as a nonlinear function of component traits, even if those traits are additively inherited. This second source of nonadditivity is easiest to exploit because average progeny merit is predicted readily.

Selection of mating pairs can be defined as simultaneous selection and mating of males and females according to predicted progeny merit. There are many ways to select and mate even a few candidates. Evaluating all possibilities becomes infeasible with many candidates; some shortcut is needed. This paper describes how linear programming can be used to select mating pairs in livestock.

THE PROBLEM OF SELECTING MATING PAIRS
The first step in selecting mating pairs is predicting merit of progeny. Consider that merit is a nonlinear function of several additively inherited traits. An index of progeny merit can be calculated by evaluating the merit function at half the arithmetic mean of sire and dam estimated breeding values for each trait (1, 4). These predictions can be organized conveniently in a matrix with rows and columns representing sires and dams.

Allaire (1) described mate selection as the choice of a mate for a preselected member of a mating pair. The best progeny in each column of that matrix would identify the appropriate sires if dams are preselected, for example. Thus, mate selection is easy to perform in the simplest case with dams preselected and no restrictions on the use of sires.

Problems arise if matings to a sire are limited by reproductive capacity, semen availability, or perhaps by a subjective constraint set by the breeder. Then matings must be chosen to maximize average merit of all selected mating pairs;
the choices cannot be made animal by animal. In addition, preselection of parents may eliminate some meritorious matings and reduce average merit of progeny.

A more general problem of selecting mating pairs is as follows: given that it is desired to select \( f \) mating pairs, select them from all possible matings between the \( M \), male, and \( F \), female, candidates so that predicted merit of progeny is maximized. Females are to be mated once at most and the \( i^{th} \) male mated no more than \( k_i \) times so \( f \leq F \) females are selected, and \( \sum_{i=1}^{M} k_i \) must be greater than or equal to \( f \).

Evaluating all mating combinations is not reasonable for more than a few candidates. For example, if all males are mated zero, or \( k_0 \), times so that \( m \) are selected, then there are \( \binom{M}{m} \binom{F}{f} / (k_0)!^m \) possible combinations. With \( F = M = 10, f = 8, \) and \( m = 4 \), that is \( \binom{10}{4} \binom{10}{8} / (2!)^4 \) or \( 2 \times 10^7 \) possibilities.

Linear programming (LP) can be used to consider all these possibilities implicitly while only explicitly evaluating a small subset of them.

**LINEAR PROGRAMMING FORMULATION**

The general problem of selecting mating pairs in LP terminology is as follows:

maximize:

\[
\sum_{i=1}^{M} \sum_{j=1}^{F} \pi_{ij} X_{ij}
\]

subject to:

\[
\sum_{i=1}^{M} X_{ij} \leq 1 \text{ for } j=1, F;
\]

\[
\sum_{j=1}^{F} X_{ij} \leq k_i \text{ for } i=1, M;
\]

\[
\sum_{i=1}^{M} \sum_{j=1}^{F} X_{ij} = f; \text{ and}
\]

\[
X_{ij} = 0 \text{ or } 1
\]

where \( \pi_{ij} \) is the predicted merit of a progeny from the \( i^{th} \) male mated to the \( j^{th} \) female, and \( X_{ij} \) is the number of progeny from the \( ij^{th} \) mating pair, restricted to 0 or 1.

In matrix notation the problem is:

maximize:

\[
\pi' x
\]

subject to:

\[
Ax \leq r
\]

and \( X_{ij} = 0 \) or 1

where \( A \) is the coefficient matrix and \( r \) is the right-hand side (RHS) vector comprising the limits on matings for females, males, and the total number of matings.

**SOLVING THE LINEAR PROGRAMMING PROBLEM**

Several properties of this problem allow great computational savings. Even though integer solutions are required, LP algorithms can be used in place of integer programming because the coefficient matrix has a structure that guarantees integer solutions if the RHS are integer [as shown by Gass (2), for LP transportation problems]. General LP packages can be used if the problem is small, say 20 males-by-20 females, or is not to be solved repeatedly. Otherwise the size of the problem \((M + F + 1)\) constraints by \( M \times F + M + F \) variables, including slacks) demands specialized programming.

Mate selection, with males and females preselected, closely resembles several LP problems in operations research. Mate selection can be viewed as assigning males to females so as to maximize progeny merit, just as in a "general assignment problem" persons must be assigned to tasks so as to maximize overall productivity. When the assignment is not one-to-one, i.e., each male may mate several females, the problem is identical to the "transportation problem."

Several efficient algorithms have been developed to solve these problems (2). Typically they are modifications of the LP simplex algorithm that employ special properties of the problem, such as the zero to one coefficient matrix. They generally perform simple operations on several small matrices of order number-
of-males by number-of-females. Thus, minimal computer core storage is required.

The problem of selecting mating pairs also can be solved by transportation methods. First include an artificial male to which unselected females will be mated and an artificial female for excess male mating possibilities. This allows constraints [2] and [3] to be expressed as the equalities:

\[ \sum_{j=1}^{M+1} X_{ij} = 1 \text{ for } j=1, F \]  

\[ \sum_{j=1}^{F+1} X_{ij} = k_i \text{ for } i=1, M. \]  

One additional constraint for the artificial female:

\[ \sum_{i=1}^{M+1} X_{i,F+1} = \sum_{i=1}^{M} k_i - f, \]  

or for the artificial male:

\[ \sum_{j=1}^{F+1} X_{M+1,j} = F - f, \]

one of which is redundant, completes the transportation problem. However, for constraint [4] to hold, the artificial by artificial mating 

\[(X_{M+1,F+1})\]

must be zero in the optimal solution. This can be achieved easily, considering the nature of the constraints, by setting progeny merit for all artificial matings equal to each other and greater than the maximum merit for real matings.

**EXAMPLE SELECTION AND MATING PROBLEM**

The following example illustrates how to set up a problem for solution by transportation methods. There are three males and four females available from which three matings are to be chosen. Indexes of progeny merit for each possible mating are in the first three rows and four columns in Table 1. Each female is to be mated once and each male up to two times as shown in the RHS row and column.

A row and column are added for the artificial male and female. The corresponding RHS elements are \( F - f = 4 - 3 \) for the male and \( \sum k_i - f = 6 - 3 \) for the female. Indexes of progeny merit for artificial matings are set to the maximum, 9, plus 1.

The optimal solution includes matings in boldface in Table 1. The maximum mean of progeny indexes is \((5 + 8 + 9)/3 = 7.33\).

**DISCUSSION**

We assume that situations do exist in which selecting mating pairs is desirable and we show that LP techniques can be employed to do it optimally. With methods borrowed from operations research, even large problems (e.g., 100 males \( \times \) 100 females) could be solved on microcomputers. Among other things, this means

**TABLE 1. Example selection and mating problem set up for solution by “transportation methods.”**

<table>
<thead>
<tr>
<th></th>
<th>Female</th>
<th>Male RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Male</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>7(^3)</td>
<td>5(^4)</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>4(^1)</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Female RHS</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

\(^1\) Artificial male and female.

\(^3\) RHS = Right-hand side.

\(^4\) Expected progeny merit.

\(^1\) Boldface matings are in the optimal strategy.
that these methods to build rational mating packages (provided mate selection is justified) could be used for data on dairy artificial insemination studs.

The selection of mating pairs also could be accomplished within LP farm models as suggested by Jansen and Wilton (3). However, the special procedures discussed here would no longer be useful, and the problem may become impractically large.

REFERENCES